$$\frac{d}{dt}\ln\Omega_v = -\frac{2q\,\mathrm{tg}\,\gamma}{\varkappa M^2}\frac{\partial}{\partial n}\ln p - \frac{1}{\cos\gamma}\left(\mathbf{v}\cdot\frac{1}{q}\frac{\partial\mathbf{v}}{\partial t}\right)$$

$$\mathbf{v} = \boldsymbol{\omega}/\omega,\ d/dt = \partial/\partial t + \mathbf{v}\cdot\nabla$$

In the theory of a non-steady thin compressed layer the generalized projection of the vorticity on the direction of velocity is constant along the trajectories only in the case of flow past a thin wing of small aspect ratio, unlike the stationary case /9/.

REFERENCES

1. TRUESDELL C., The kinematic of vorticity. Bloomington: Indiana Univ., Publ., Science Ser., No.19, 1954.
2. ZEITUNYAN R.K., Theory of three-dimensional vortical flows of perfect fluids. In: Numerical Methods in the Mechanics of Continua. Vol.8, No.5, Novosibirsk, VTs SO Akad. Nauk SSSR, 1977.
3. MOBBS S.D., Some vorticity theorems and conservation laws for non-barotropic fluids. J. Fluid Mech., Vol.108, 1981.
4. GOLUBINSKII A.I., On the conservation of the generalized circulation of velocity in steady flows of an ideal gas. Dokl. Akad. Nauk SSSR, Vol.196, No.5, 1971.
5. GOLUBINSKII A.I. and SYCHEV VIK.V., On certain properties of invariance of vortical gas flows. Dokl. Akad. Nauk SSSR, Vol.237, No.4, 1977.
6. VON MISES R., Mathematical Theory of Compressible Fluid Flows. N.Y. Acad. Press, 1958.
7. CHERNYI G.G., Hypersonic Gas Flows. Moscow, Fizmatgiz, 1959.
8. GOLUBINSKII A.I. and GOLUBKIN V.N., On three-dimensional hypersonic gas flow past a thin wing. Dokl. Akad. Nauk SSSR, Vol.234, No.5, 1977.
9. GOLUBINSKII A.I. and GOLUBKIN V.N., Three-dimensional hypersonic flow past a body of finite thickness. Uch. zap. TsAGI, Vol.13, No.2, 1982.
10. GOLUBINSKII A.I. and GOLUBKIN V.N., On hypersonic non-equilibrium gas flow past a wing of small aspect ratio. Izv. Akad. Nauk SSSR, MZhG, No.6, 1983.
11. GOLUBINSKII A.I. and GOLUBKIN V.N., Three-dimensional hypersonic radiating gas flow past a wing. PMTF, Vol.48, No.3, 1983.
12. GOLUBKIN V.N., Hypersonic gas flow pas a triangular wing at large angles at attack. PMM Vol.48, No.3, 1984.
13. GUDERLEY K.G., Theory of Transonic Flows. Oxford, N.Y., Pergamon Press Ltd., 1962.
14. ESHLI KH. and LENDAL M., Aerodynamics of Wings and Bodies of Airplanes. Moscow, Mashinostroenie, 1969.

Translated by L.K.

PMM U.S.S.R.,Vol.49,No.1,pp.88-95,1985
Printed in Great Britain

0021-8928/85 $10.00+0.00

COMPATIBILITY EQUATIONS, STRESS FUNCTIONS, AND VARIATIONAL PRINCIPLES IN THE THEORY OF PRESTRESSED SHELLS*

L.M. ZUBOV

General statements of the theory of small deformations of thin shells with initial stresses are considered /1/. Compatibility equations are derived for the kinematic quantities, functions are found that satisfy the equilibrium equations identically, different variational principles of statics are formulated and proved, and distortion boundary conditions are obtained. The presence of initial stresses induces substantial singularities into these sections of the theory as compared with the linear theory of unstressed shells /2—5/. These singularities are due to the fact that the specific potential energy in the theory of small deformations of elastic shells with initial stresses depends not only on the tensors governing the change in metric and curvature of the surface, but also the rotation vector /1/.

The results obtained can be applied in shell stability problems as well as in the analysis of large shell deformations by the method of successive loadings when a linear problem of small deformations measured

**Prikl.Matem.Mekhan.*,49,1,120-129,1985

from the state of stress corresponding to the previous step is solved in each step of the calculation process.

1. The system of equations describing small deformations of thin elastic shells when there are initial stresses consists /1/ of the equilibrium equation for the force quantities

$$\nabla\cdot[\mathbf{H}-\mathbf{M}\cdot\mathbf{B}+\mathbf{G}\cdot(\nabla\cdot\mathbf{M})\,\mathbf{N}]+\mathbf{f}+\nabla\cdot(\boldsymbol{\mu}\mathbf{N})=0 \tag{1.1}$$

$$\mathbf{H}=\mathbf{K}+\frac{1}{2}\gamma\mathbf{e}+\boldsymbol{\lambda}\mathbf{N}\quad(\boldsymbol{\lambda}\cdot\mathbf{N}=0)$$

governing the relationships connecting the force and kinematic quantities

$$\mathbf{H}=\frac{\partial a}{\partial\mathbf{F}},\quad \mathbf{K}=\frac{\partial a}{\partial\boldsymbol{\varepsilon}},\quad \gamma=\frac{\partial a}{\partial\chi},\quad \boldsymbol{\lambda}=\frac{\partial a}{\partial\boldsymbol{\vartheta}} \tag{1.2}$$

$$\mathbf{M}=-\partial a/\partial\boldsymbol{\varkappa},\ a=a\,(\boldsymbol{\varepsilon},\boldsymbol{\varkappa},\boldsymbol{\vartheta},\chi)$$

and the formulas expressing the kinematic quantities in terms of the displacement vector field $\mathbf{w}$ of the middle surface of the shell

$$\mathbf{F}=\nabla\mathbf{w}=\boldsymbol{\varepsilon}+\chi\mathbf{e}+\boldsymbol{\vartheta}\mathbf{N},\ \boldsymbol{\vartheta}=\mathbf{N}\cdot\mathbf{F}^T=\nabla w+\mathbf{B}\cdot\mathbf{u} \tag{1.3}$$

$$\boldsymbol{\varepsilon}=\frac{1}{2}\,[(\nabla\mathbf{u})\cdot\mathbf{G}+\mathbf{G}\cdot(\nabla\mathbf{u})^T]-\mathbf{B}w,\quad \chi=\frac{1}{2}\,\mathbf{N}\cdot(\nabla\times\mathbf{u})$$

$$\boldsymbol{\varkappa}=\boldsymbol{\varkappa}^T=(\nabla\boldsymbol{\vartheta})\cdot\mathbf{G}+\mathbf{B}\cdot\mathbf{F}^T=\frac{1}{2}\,[(\nabla\boldsymbol{\omega})\cdot\mathbf{e}-\mathbf{e}\cdot(\nabla\boldsymbol{\omega})^T]+$$

$$\frac{1}{2}(\boldsymbol{\varepsilon}\cdot\mathbf{B}+\mathbf{B}\cdot\boldsymbol{\varepsilon}),\quad \boldsymbol{\omega}=\boldsymbol{\vartheta}\times\mathbf{N}+\chi\mathbf{N},\quad \mathbf{G}=\mathbf{E}-\mathbf{N}\mathbf{N}$$

$$\mathbf{e}=-\mathbf{e}^T=-\mathbf{G}\times\mathbf{N},\ \mathbf{B}=-\nabla\mathbf{N},\ \mathbf{u}=\mathbf{w}\cdot\mathbf{G},\ w=\mathbf{w}\cdot\mathbf{N}$$

The $\mathbf{N}$ in (1.1)–(1.3) is the unit vector normal to the middle surface O of the shell, $\mathbf{G}$ and $\mathbf{B}$ are the first and second fundamental surface tensors, $\mathbf{e}$ is the discriminant tensor, $\mathbf{E}$ is the unit tensor, ∇ is the nabla operator on the surface, $\boldsymbol{\varepsilon}$ is the linear surface strain tensor, $\boldsymbol{\varkappa}$ is the tensor of curvature variation, $\boldsymbol{\omega}$ is the linear vector of surface rotation, a is the specific potential energy of a prestressed shell (per unit middle surface area), and $\mathbf{f}$ and $\boldsymbol{\mu}\times\mathbf{N}$ $(\boldsymbol{\mu}\cdot\mathbf{N}=0)$ are the intensities of the additional force and moment loads distributed over the surface O. The vector fields $\mathbf{f}$ and $\boldsymbol{\mu}$ are considered given, i.e., are independent of the displacement vector $\mathbf{w}$ and its derivatives.

In the case of small initial strain and a membrane initial state of stress, the governing relationships are written as follows /1/:

$$\mathbf{K}=\frac{Eh}{1-\nu^2}\,[(1-\nu)\,\boldsymbol{\varepsilon}+\nu\mathbf{G}\,\mathrm{tr}\,\boldsymbol{\varepsilon}]+\frac{1}{2}\chi\,(\mathbf{S}\cdot\mathbf{e}-\mathbf{e}\cdot\mathbf{S}) \tag{1.4}$$

$$\mathbf{M}=-\frac{Eh^3}{12\,(1-\nu^2)}\,[(1-\nu)\,\boldsymbol{\varkappa}+\nu\mathbf{G}\,\mathrm{tr}\,\boldsymbol{\varkappa}],\quad \boldsymbol{\lambda}=\mathbf{S}\cdot\boldsymbol{\vartheta}$$

$$\gamma=\chi\,\mathrm{tr}\,\mathbf{S}+\mathrm{tr}\,(\mathbf{S}\cdot\mathbf{e}\cdot\boldsymbol{\varepsilon})$$

$$a=\frac{Eh}{2\,(1-\nu^2)}\,[\mathrm{tr}^2\,\boldsymbol{\varepsilon}-2\,(1-\nu)\det\boldsymbol{\varepsilon}]+\frac{Eh^3}{24\,(1-\nu^2)}\times$$

$$[\mathrm{tr}^2\,\boldsymbol{\varkappa}-2\,(1-\nu)\det\boldsymbol{\varkappa}]+\chi\,\mathrm{tr}\,(\mathbf{S}\cdot\mathbf{e}\cdot\boldsymbol{\varepsilon})+\frac{1}{2}\chi^2\,\mathrm{tr}\,\mathbf{S}+\frac{1}{2}\,\boldsymbol{\vartheta}\cdot\mathbf{S}\cdot\boldsymbol{\vartheta}$$

Here E is Young's modulus, ν is Poisson's ratio, h is the shell thickness, and $\mathbf{S}$ is the tensor of the initial forces. The governing relationships for an arbitrary initial state are presented in /1/.

The force boundary conditions obtained in /1/ on the shell contour can be written as follows:

$$\mathbf{m}\cdot[\mathbf{H}+(\nabla\cdot\mathbf{M})\,\mathbf{N}-\mathbf{M}\cdot\mathbf{B}]+\frac{\partial}{\partial s}\,(M_{mt}\mathbf{N})= \tag{1.5}$$

$$\mathbf{l}+\frac{\partial}{\partial s}\,(d_t\mathbf{N})-(\mathbf{m}\cdot\boldsymbol{\mu})\,\mathbf{N},\quad M_{mm}=d_m$$

$$(\mathbf{d}=d_m\mathbf{m}+d_t\mathbf{t},\ \mathbf{m}\cdot\mathbf{M}=M_{mm}\mathbf{m}+M_{mt}\mathbf{t})$$

where $\mathbf{m},\mathbf{t}$ are unit vectors of the normal and tangent to the contour $\Gamma\,(\mathbf{m}\cdot\mathbf{N}=0)$, s is the running length of the arc of the contour, $\mathbf{l}$ is the intensity of the external load distributed over the contour $\mathbf{d}\times\mathbf{N}$ $(\mathbf{d}\cdot\mathbf{N}=0)$ is the intensity of the moment distributed over the boundary curve.

The geometric boundary conditions consist of specifying the displacement vector $\mathbf{w}$ and the components of the angle of rotation $\vartheta_m=\mathbf{m}\cdot\boldsymbol{\vartheta}$ on Γ.

2. If (1.3) is substituted into (1.2), and then into (1.1), a system of three equations is obtained in the three displacement vector components. In addition to this, the force quantities $\mathbf{H},\mathbf{M}$ can be taken as fundamental unknowns. The compatibility equations imposed on the kinematic quantities $\mathbf{F},\boldsymbol{\varkappa}$ and representing the result of eliminating the displacements from the relationships (1.3) will be the equations to define them along with the equilibrium

equations (1.1). The governing relationships (1.2) should here be inverted, i.e., the kinematic quantities should be expressed in terms of the force quantities.

The necessary and sufficient condition that the tensor field $\mathbf{F} = \mathbf{G}\cdot\mathbf{F}$ be the gradient of a certain vector field $\mathbf{w}$ on the surface is the equation /6, 7/

$$\nabla\cdot(\mathbf{e}\cdot\mathbf{F}) = 0 \tag{2.1}$$

The relationship

$$\varkappa - [\nabla(\mathbf{N}\cdot\mathbf{F}^T)]\cdot\mathbf{G} - \mathbf{B}\cdot\mathbf{F}^T = 0 \tag{2.2}$$

results from (1.3).

Equations (2.1) and (2.2) are equivalent to six scalar relationships for the nine components of the tensors $\mathbf{F}$ and $\varkappa$, and are the compatibility equations for the kinematic quantities. When they are satisfied, the displacement vector is determined by the quadrature

$$\mathbf{w} = \int d\mathbf{R}\cdot\mathbf{F}$$

where $\mathbf{R}$ is the radius-vector of a surface point. In a simply-connected domain it is sufficient to give the displacement of some point of the surface for a single-valued determination of $\mathbf{w}$. In the case of a domain not simply-connected (a shell with holes) the vector $\mathbf{w}$ is not generally single-valued. If slits transforming the shell surface into a simply-connected domain are made, then the vector $\mathbf{w}$ can undergo a discontinuity of the first kind on intersecting the slits, where the magnitude of the jump in the vector is identical at all points of each of the slits.

Since the number of compatibility equations does not agree with the number of equilibrium equations, the static-geometric analogy does not generally hold in the theory of prestressed shells. This analogy is valid in the special case of membrane shell theory with initial stresses (in the case it is necessary to set $\mathbf{M} = 0$ and to eliminate the tensor $\varkappa$ from consideration) and is a consequence of the analogy noted in /6/.

The geometric boundary conditions on the part Γ_1 of the shell edge $\mathbf{w} = \mathbf{w}^*$, $\vartheta_m = \vartheta_m^*$, where the asterisk denotes given functions of the coordinate s, can be replaced by conditions on the boundary values of the tensor $\mathbf{F}$. Since $\mathbf{t}\cdot\mathbf{F} = \partial\mathbf{w}/\partial s$ on Γ_1, we have

$$\mathbf{t}\cdot\mathbf{F} = \partial\mathbf{w}^*/\partial s,\ \mathbf{m}\cdot\mathbf{F}\cdot\mathbf{N} = \vartheta_m^* \tag{2.3}$$

Since the tensor-gradient of the displacement vector is often called the distortion tensor (see /8/, say), the boundary conditions (2.3) can be called distortion conditions.

The distortion conditions were obtained /5/ as an intermediate result in deriving the strain conditions. The distortion conditions are not utilized in the linear theory of shells without initial stresses. The role of the strain conditions is taken over by the distortion conditions in the theory of prestressed shells.

If the curve Γ_1 consists of a simply-connected section, then the geometric boundary conditions are restored by the distortion conditions apart from an arbitrary constant vector by quadratures. Since arbitrary translation (translational displacement) of a shell is not essential in an equilibrium problem, in the case of a connected curve Γ_1 the geometric and distortion conditions can be considered to be equivalent. If Γ_1 consists of separate disconnected sections, the distortion boundary conditions are not sufficient to restore the geometric conditions (and therefore, to formulate the boundary value problem also). It is still required to give relations governing the mutual translational displacement of the separate sections of the curve Γ_1.

3. Starting from the identity /7/ that is valid for an arbitrary twice differentiable vector field $\mathbf{a}$

$$\mathbf{N}\cdot[\nabla\times(\nabla\mathbf{a})] = \nabla\cdot(\mathbf{e}\cdot\nabla\mathbf{a}) = 0 \tag{3.1}$$

it can be seen that the general solution of the equilibrium equations (1.1) can be represented in the form

$$\mathbf{H} = \mathbf{e}\cdot\nabla\mathbf{\Phi} + \mathbf{\Psi}\cdot\mathbf{B} - \mathbf{G}\cdot(\nabla\cdot\mathbf{\Psi})\,\mathbf{N} + \mathbf{H}',\ \mathbf{M} = \mathbf{\Psi} + \mathbf{M}' \tag{3.2}$$

where $\mathbf{\Phi}$ is an arbitrary twice-differentiable vector field, and $\mathbf{\Psi}$ is an arbitrary twice-differentiable symmetric tensor satisfying the condition $\mathbf{N}\cdot\mathbf{\Psi} = 0$. The prime in (3.2) denotes a certain particular solution of (1.1) corresponding to the surface loads $\mathbf{f}$, $\boldsymbol{\mu}$. We call the components $\mathbf{\Phi}$ and $\mathbf{\Psi}$ stress functions. Relations (3.2) express nine scalar force quantities, the components of the tensors $\mathbf{H}$ and $\mathbf{M}$, in terms of six stress functions; the equilibrium equations are here satisfied identically.

The force boundary conditions (1.5) are written as follows in terms of the stress functions

$$\frac{\partial\mathbf{\Phi}}{\partial s} + \frac{\partial}{\partial s}(\psi_{mt}\mathbf{N}) + \mathbf{L}' = \mathbf{l} + \frac{\partial}{\partial s}(d_t\mathbf{N}) - (\mathbf{m}\cdot\boldsymbol{\mu})\,\mathbf{N} \tag{3.3}$$

$$\psi_{mm} + M'_{mm} = d_m, \quad \mathbf{m}\cdot\Psi' \equiv \psi_{mm}\mathbf{m} + \psi_{mt}\mathbf{t}$$
$$\mathbf{L}' \equiv \mathbf{m}\cdot\mathbf{H}' + \mathbf{m}\cdot[\nabla\cdot(\mathbf{M}'\mathbf{N})] + \frac{\partial}{\partial s}(M'_{mt}\mathbf{N})$$

4. We introduce a specific additional energy A into the considerations, as a function of the static quantities $\mathbf{H}, \mathbf{M}$ related by a Legendre transformation to the specific potential energy a. By the property of the Legendre transformation we have

$$\mathbf{F} = \frac{\partial A}{\partial \mathbf{H}}, \quad \varepsilon = \frac{\partial A}{\partial \mathbf{K}}, \quad \vartheta = \frac{\partial A}{\partial \lambda}, \quad \chi = \frac{\partial A}{\partial \gamma} \tag{4.1}$$
$$\varkappa = -\partial A/\partial \mathbf{M}, \quad A = A(\mathbf{H}, \mathbf{M})$$

For the case of a low initial strain and a membrane initial state of stress the function A is calculated from (1.4) and has the form

$$2A = Eh[(1+\nu)\operatorname{tr}\overline{\mathbf{K}}^2 - \nu\operatorname{tr}^2\overline{\mathbf{K}}] + \frac{12}{Eh^3}[(1+\nu)\operatorname{tr}\mathbf{M}^2 - \nu\operatorname{tr}^2\mathbf{M}] + \lambda\cdot\sigma\cdot\lambda + \tag{4.2}$$
$$Eh\left[\operatorname{tr}\overline{\mathbf{S}} + \frac{1}{2}(1+\nu)\operatorname{tr}^2\overline{\mathbf{S}} - (1+\nu)\operatorname{tr}\overline{\mathbf{S}}^2\right]^{-1}[\overline{\gamma} - (1+\nu)\operatorname{tr}(\overline{\mathbf{K}}\cdot\overline{\mathbf{S}}\cdot\mathbf{e})]^2$$
$$\mathbf{K} = Eh\overline{\mathbf{K}}, \quad \mathbf{S} = Eh\overline{\mathbf{S}}, \quad \gamma = Eh\overline{\gamma}, \quad \sigma\cdot\mathbf{S} = \mathbf{G}, \quad \mathbf{N}\cdot\sigma = \sigma\cdot\mathbf{N} = 0$$

Here σ is a two-dimensional tensor inverse to the initial force tensor.

5. The relationships obtained above enable us to formulate variational principles for the theory of prestressed shells that are analogous to the principles set down in /9/ for a three-dimensional elastic medium. We assume that the shell boundary consists of two parts Γ_1 and Γ_2. Geometric conditions are given on Γ_1 and force conditions on Γ_2. We consider the following functionals

$$U_1(\mathbf{w}) = \iint_O a[\mathbf{F}(\mathbf{w}), \varkappa(\mathbf{w})]\,dO - \iint_O [\mathbf{f}\cdot\mathbf{w} - \mu\cdot\vartheta(\mathbf{w})]\,dO - \tag{5.1}$$
$$\int_{\Gamma_2}\left\{\left[\mathbf{l} + \frac{\partial}{\partial s}(\mathbf{m}\times\mathbf{d})\right]\cdot\mathbf{w} - \mathbf{d}\cdot\vartheta(\mathbf{w})\right\}ds$$
$$U_2[\mathbf{w}, \mathbf{H}, \mathbf{M}] = \iint_O [\mathbf{H}\cdot\cdot\mathbf{F}^T(\mathbf{w}) - \mathbf{M}\cdot\cdot\varkappa(\mathbf{w}) - \tag{5.2}$$
$$A(\mathbf{H}, \mathbf{M}) - \mathbf{f}\cdot\mathbf{w} + \mu\cdot\vartheta(\mathbf{w})]\,dO - J$$
$$J = \int_{\Gamma_1}\left\{\left[\mathbf{m}\cdot(\mathbf{H} + \nabla\cdot(\mathbf{M}\mathbf{N})) - \mathbf{m}\cdot\mathbf{B}M_{mm} + \right.\right.$$
$$\left.\left.\frac{\partial}{\partial s}(M_{mt}\mathbf{N})\right]\cdot(\mathbf{w} - \mathbf{w}^*) - M_{mm}[\vartheta_m(\mathbf{w}) - \vartheta_m^*]\right\}ds -$$
$$\int_{\Gamma_2}\left\{\left[\mathbf{l} + \frac{\partial}{\partial s}(\mathbf{m}\times\mathbf{d})\right]\cdot\mathbf{w} - \mathbf{d}\cdot\vartheta(\mathbf{w})\right\}ds$$
$$P\cdot\cdot Q \equiv \operatorname{tr}(P\cdot Q)$$
$$U_3[\mathbf{w}, \mathbf{F}, \varkappa, \mathbf{H}, \mathbf{M}] = \iint_O \{a(\mathbf{F}, \varkappa) - \mathbf{H}^T\cdot\cdot(\mathbf{F} - \nabla\mathbf{w}) + \tag{5.3}$$
$$\mathbf{M}\cdot\cdot[\varkappa - \nabla(\nabla w + \mathbf{B}\cdot\mathbf{w})\cdot\mathbf{G} - \mathbf{B}\cdot(\nabla\mathbf{w})^T] -$$
$$\mathbf{f}\cdot\mathbf{w} + \mu\cdot\vartheta(\mathbf{w})\}\,ds - J$$

The functional U_1 is defined on a set of fairly smooth displacement fields satisfying geometric conditions on Γ_1. The displacement fields and force quantities not subjected to any bundary conditions vary independently in the functional (5.2). The variable functions in the functional (5.3) are the displacements, and the kinematic and force quantities which are also not connected by additional conditions. The equilibrium equations in displacements and the force boundary conditions result form the stationarity of the functional U_1. The equilibrium equations (1.1), relations (4.1), and the geometric and force conditions result from the condition $\delta U_2 = 0$. The requirement $\delta U_3 = 0$ is equivlaent to Eqs. (1.1), relations (1.2) and (1.3), and also the force and geometric boundary conditions.

The variational theorems with the functionals (5.1)–(5.3) are analogous to the Lagrange, Reissner, and Hu-Washizu conditions in the theory of elasticity /10/. It is easy to present a formulation of these principles (we shall not do this here) even for different cases of combined boundary conditions: a hinged support, a moving hinge, a sliding frame, etc.

The formulation of the above-mentioned variational principles can be extended to the case of a follower pressure of intensity p, distributed uniformly over the surface O. If the conservation conditions for a hydrostatic load set up in /1/ are satisfied, it is sufficient to append the potential for this load /1/ to the expressions for the functionals (5.1)–(5.3)

$$\Pi = \frac{1}{2} p \iint_O (\boldsymbol{\vartheta} \cdot \mathbf{u} - w \operatorname{tr} \boldsymbol{\varepsilon})\, dO$$

Expressing the force quantities according to (3.2) in terms of the stress functions, we consider the following functional over the stress functions

$$V_1[\boldsymbol{\Phi}, \boldsymbol{\Psi}] = \iint_O A(\boldsymbol{\Phi}, \boldsymbol{\Psi})\, dO - \int_{\Gamma_1} \left\{ \mathbf{w}^* \cdot \left[\frac{\partial \boldsymbol{\Phi}}{\partial s} + \frac{\partial}{\partial s} (\mathbf{N} \psi_{mt}) \right] - \vartheta_m^* \psi_{mm} \right\} ds \tag{5.4}$$

The functional V_1 is defined on a set of twice-differentiable vector $\boldsymbol{\Phi}$ and tensor $\boldsymbol{\Psi}$ fields subjected to the boundary conditions (3.3) on Γ_2. Note that the value of the functional V_1 does not change if $\boldsymbol{\Phi}$ is replaced by $\boldsymbol{\Phi} + \mathbf{c}$, where $\mathbf{c}$ is an arbitrary constant vector.

We will show that the stationarity condition for the functional V_1 is equivalent to the compatibility equations (2.1) and (2.2) written in terms of the stress functions and the geometric boundary conditions on Γ_1. On the basis of (4.1), we write the variation of the functional V_1 after integration by parts in the form

$$\begin{aligned} &\iint_O (\nabla \cdot \mathbf{e} \cdot \mathbf{F}) \cdot \delta \boldsymbol{\Phi} + [(\mathbf{B} \cdot \mathbf{F}^T + \nabla \boldsymbol{\vartheta} - \boldsymbol{\varkappa}) \cdot \cdot \delta \boldsymbol{\Psi}]\, dO - \\ &\oint_\Gamma (\mathbf{t} \cdot \mathbf{F} \cdot \delta \boldsymbol{\Phi} + \vartheta_t \delta \psi_{mt} + \vartheta_m \delta \psi_{mm})\, ds - \\ &\int_{\Gamma_1} \left\{ \mathbf{w}^* \cdot \left[\frac{\partial \delta \boldsymbol{\Phi}}{\partial s} + \frac{\partial}{\partial s} (\mathbf{N} \delta \psi_{mt}) \right] - \vartheta_m^* \delta \psi_{mm} \right\} ds \end{aligned} \tag{5.5}$$

According to (3.3) the possible variations in the stress functions on the curve Γ_2 should be subject to the condition

$$\partial/\partial s\, (\delta \boldsymbol{\Phi} + \mathbf{N} \delta \psi_{mt}) = 0, \quad \delta \psi_{mm} = 0 \tag{5.6}$$

Let $\delta V_1 = 0$. First setting $\delta \boldsymbol{\Phi} = \delta \boldsymbol{\Psi} = 0$ on Γ (this is compatible with the constraint (5.6)), by virtue of the fundamental lemma of the calculus of variations we arrive at the compatibility equations (2.1) and (2.2). These equations mean that a vector field $\mathbf{w}$ exists whose surface gradient is $\mathbf{F}(\boldsymbol{\Phi}, \boldsymbol{\Psi})$. In a simply-connected domain the vector $\mathbf{w}$ is a single-valued function of the coordinates on the surface defined apart from an additive vector constant. Taking this into account, we transform the stationarity condition of the functional

$$\begin{aligned} &- \oint_\Gamma \left[\frac{\partial}{\partial s} (\mathbf{w} \cdot \delta \boldsymbol{\Phi}) + \frac{\partial}{\partial s} (\mathbf{w} \cdot \mathbf{N} \delta \psi_{mt}) \right] ds + \\ &\oint_\Gamma \left[\mathbf{w} \cdot \frac{\partial}{\partial s} (\delta \boldsymbol{\Phi} + \mathbf{N} \delta \psi_{mt}) - \mathbf{m} \cdot (\nabla w + \mathbf{B} \cdot \mathbf{w})\, \delta \psi_{mm} \right] ds - \\ &\int_{\Gamma_1} \left[\mathbf{w}^* \cdot \frac{\partial}{\partial s} (\delta \boldsymbol{\Phi} + \mathbf{N} \delta \psi_{mt}) - \vartheta_m^* \delta \psi_{mm} \right] ds = 0 \end{aligned} \tag{5.7}$$

The first integral in (5.7) is obviously zero. Since $\Gamma = \Gamma_1 \cup \Gamma_2$, and conditions (5.6) hold on Γ_2, we conclude from the arbitrariness of $\delta \boldsymbol{\Phi}, \delta \boldsymbol{\Psi}$ that the conditions $\mathbf{m} \cdot (\nabla w + \mathbf{B} \cdot \mathbf{w}) = \vartheta_m^*, \mathbf{w} = \mathbf{w}^*$ are satisfied on Γ_1. We note that the above-mentioned indeterminacy in the form of the vector constant is eliminated by this last relationship.

If the shell surface is not simply-connected, then by transforming it into a simply-connected domain by making the requisite number of slits (partitions), it can be shown that the condition that the displacements are single-valued also results from the stationarity of the functional V_1 in the case of a multiconnected domain.

The force boundary conditions (3.3) are an ordinary differential equation in the vector $\boldsymbol{\Phi} + \mathbf{N} \psi_{mt}$. Since the stress vector function $\boldsymbol{\Phi}$ is defined to within an arbitrary constant vector, if Γ_2 consists of one connected section, the force conditions can be satisfied without loss of generality by setting

$$\boldsymbol{\Phi} + \mathbf{N} \psi_{mt} = \boldsymbol{\varphi}^*, \quad \psi_{mm} = d_m - M_{mm}' \tag{5.8}$$

on Γ_2 where $\boldsymbol{\varphi}^*$ is a certain particular solution of the equation

$$\frac{d\boldsymbol{\varphi}}{ds} = \mathbf{l} - \mathbf{L}' + \frac{d}{ds} (\mathbf{m} \times \mathbf{d}) - (\mathbf{m} \cdot \boldsymbol{\mu})\, \mathbf{N}$$

We set

$$V_1'[\boldsymbol{\Phi}, \boldsymbol{\Psi}] = \iint_O A(\boldsymbol{\Phi}, \boldsymbol{\Psi})\, dO + \int_{\Gamma_1} \left[\frac{\partial \mathbf{w}^*}{\partial s} \cdot (\boldsymbol{\Phi} + \mathbf{N} \psi_{mt}) + \vartheta_m^* \psi_{mm} \right] ds \tag{5.9}$$

It can be verified that the stationarity condition for the functional V_1' over the stress functions satisfying the condition (5.8) on Γ_2 is equivalent to the compatibility conditions (2.1), (2.2) and the distortion boundary conditions on Γ_1.

The strain boundary conditions in classical linear shell theory are derived from an additional energy variational principle in /11/.

Both the distortion conditions and the force conditions in the form (5.8) are natural boundary conditions for the functionals V_2 and V_3 presented below

$$V_2[\mathbf{F},\ \boldsymbol{\varkappa},\ \boldsymbol{\Phi},\ \boldsymbol{\Psi}] = \iint_O \{[\mathbf{e}\cdot\nabla\boldsymbol{\Phi} + \boldsymbol{\Psi}\cdot\mathbf{B} - \mathbf{G}\cdot(\nabla\cdot\boldsymbol{\Psi})\,\mathbf{N} + \mathbf{H}']\cdot\cdot\mathbf{F}^T - (\boldsymbol{\Psi} + \mathbf{M}')\cdot\cdot\boldsymbol{\varkappa} - a(\mathbf{F},\ \boldsymbol{\varkappa})\}\,dO + W \tag{5.10}$$

$$W = \int_{\Gamma_1} \left[\frac{\partial \mathbf{w}^*}{\partial s}\cdot(\boldsymbol{\Phi} + \mathbf{N}\psi_{mt}) + \vartheta_m^*\psi_{mm}\right] ds + \int_{\Gamma_2} [\mathbf{t}\cdot\mathbf{F}\cdot(\boldsymbol{\Phi} + \mathbf{N}\psi_{mt} - \boldsymbol{\varphi}^*) + \vartheta_m(\psi_{mm} - d_m + M'_{mm})]\,ds$$

$$V_3[\mathbf{F},\ \boldsymbol{\varkappa},\ \mathbf{H},\ \mathbf{M},\ \boldsymbol{\Phi},\ \boldsymbol{\Psi}] = \iint_O \{A(\mathbf{H},\ \mathbf{M}) - \mathbf{F}^T\cdot\cdot[\mathbf{H} - \mathbf{e}\cdot\nabla\boldsymbol{\Phi} - \boldsymbol{\Psi}\cdot\mathbf{B} + \mathbf{G}\cdot(\nabla\cdot\boldsymbol{\Psi})\,\mathbf{N} - \mathbf{H}'] + \boldsymbol{\varkappa}\cdot\cdot(\mathbf{M} - \boldsymbol{\Psi} - \mathbf{M}')\}\,dO + W \tag{5.11}$$

The necessary and sufficient conditions for the functional (5.10) to be stationary are Eqs.(2.1), (2.2), the governing relations (1.2) in which the force quantities are expressed in terms of the stress functions, and the boundary conditions (2.3) and (5.8). The requirement for the functional (5.11) to be stationary is equivalent to the compatibility equations, the relations (4.1) and (3.2), and also the distortion and force boundary conditions.

We will present still another formulation of the Clapeyron theorem for prestressed shells that results from (1.1), (1.2), and (1.5)

$$\frac{1}{2}\oint_\Gamma \left\{\left[\mathbf{l} + \frac{\partial}{\partial s}(d_t\mathbf{N})\right]\cdot\mathbf{w} - \mathbf{d}\cdot\boldsymbol{\vartheta}\right\} ds + \frac{1}{2}\iint_O (\mathbf{f}\cdot\mathbf{w} - \boldsymbol{\mu}\cdot\boldsymbol{\vartheta})\,dO = \frac{1}{2}\iint_O (\mathbf{H}\cdot\cdot\mathbf{F}^T - \mathbf{M}\cdot\cdot\boldsymbol{\varkappa})\,dO = \iint_O a\,dO$$

6. The general case of an initial state of stress for a shell when inversion of the governing relations (1.2) is possible was considered above. Moreover, exceptional cases exist for which it is impossible to express all the kinematic quantities in terms of the forces, which results in a modification of the number of statements in the theory. The most important of such exmaples is a cylindrical shell (of arbitrary section) subjected to uniaxial preliminary tension or compression in the direction of the cylinder generators. We take the length of the arc of the shell transverse section outline as the x_1 coordinate on the cylinder surface, and the distance measured from the cylinder axis as the x_2 coordinate. The unit vectors tangent to the coordinate lines will be denoted by $\mathbf{e}_1$, $\mathbf{e}_2$. The tensor of the initial forces has the form $\mathbf{S} = T\mathbf{e}_2\mathbf{e}_2$ in this case. Consequently, as is seen from (1.4), the specific potential energy a is independent of the component $\vartheta_1 = \mathbf{e}_1\cdot\boldsymbol{\vartheta}$ of the rotation vector. Therefore, this component be expressed in terms of the force quantities. In compiling the system of equations in the force quantities by a natural path we arrive at the problem of determining the surface displacement vector by means of a given field of kinematic quantities $\boldsymbol{\varepsilon}$, $\varkappa$, $\boldsymbol{\varkappa}$, $\vartheta_2 = \mathbf{e}_2\cdot\boldsymbol{\vartheta}$, and the derivation of the corresponding compatibility conditions. These conditions consist of five equations and have the form

$$\begin{gathered}\frac{\partial\varkappa_{11}}{\partial x_2} - k\frac{\partial F_{11}}{\partial x_2} - \frac{\partial\varkappa_{21}}{\partial x_1} = 0\\ \varkappa_{22} - \frac{\partial\vartheta_2}{\partial x_2} = 0,\quad \frac{\partial F_{11}}{\partial x_2} - \frac{\partial F_{21}}{\partial x_1} + k\vartheta_2 = 0\\ \frac{\partial F_{12}}{\partial x_2} - \frac{\partial F_{22}}{\partial x_1} = 0,\quad \varkappa_{21} - kF_{21} - \frac{\partial\vartheta_2}{\partial x_1} = 0\\ k = k(x_1) = -\mathbf{e}_1\cdot\frac{\partial\mathbf{N}}{\partial x_1}\\ \varkappa_{\alpha\beta} = \mathbf{e}_\alpha\cdot\boldsymbol{\varkappa}\cdot\mathbf{e}_\beta,\quad F_{\alpha\beta} = \mathbf{e}_\alpha\cdot\mathbf{F}\cdot\mathbf{e}_\beta\quad (\alpha,\ \beta = 1,\ 2)\end{gathered} \tag{6.1}$$

If (6.1) are satisfied, the shell displacements are determined by quadratures apart from an arbitrary translation and an arbitrary rotation around the $\mathbf{e}_2$ axis.

In finding the general solution of the equilibrium equations (1.1) it must be taken into account that the equation $\lambda_1 = \mathbf{e}_1\cdot\mathbf{H}\cdot\mathbf{N} = 0$ holds in the singular case under consideration. This relationship imposes the following constraint on the functions $\boldsymbol{\Phi}$, $\boldsymbol{\Psi}$ in (3.2)

$$\frac{\partial\Phi}{\partial x_2} - \frac{\partial\psi_{11}}{\partial x_1} - \frac{\partial\psi_{12}}{\partial x_2} = 0,\quad \Phi = \boldsymbol{\Phi}\cdot\mathbf{N},\quad \psi_{\alpha\beta} = \mathbf{e}_\alpha\cdot\boldsymbol{\Psi}\cdot\mathbf{e}_\beta \tag{6.2}$$

Condition (6.2) can be satisfied by setting

$$\psi_{12} = \Phi - \frac{\partial\psi}{\partial x_1}, \quad \psi_{11} = \frac{\partial\psi}{\partial x_2}$$

where ψ is an arbitrary twice-differentiable function.

Therefore, for uniaxial prestressing the general solution of the equilibrium equations contains five stress functions $\Phi, \mathbf{e}_\alpha\cdot\mathbf{\Phi}$ $(\alpha = 1, 2), \psi, \psi_{22}$, unlike the general case. It can be verified that even in this case the compatibility equations (6.1) result from a variational principle with a functional of type V_1.

If there are no initial stresses in the shell, then the stress functions $\mathbf{\Phi}, \mathbf{\Psi}$ should be subjected to the conditions $\lambda = 0, \gamma = 0$. By starting from this it is possible to arrive at the three stress functions of linear shell theory /3—5/ in terms of which the general solution of the three equilibrium equations for the symmetric force and moment tensors is expressed.

7. As an application we consider a new energy criterion for the buckling of thin plates that results from the principle of complementary work. The equations $\mathbf{K} = \gamma = 0$ are satisfied for the bending strains of a slab and we have $\mathbf{\Phi}\cdot\mathbf{G} = 0$ from (3.2). On the basis of (3.2), (4.2) and (5.9), the stability energy criterion for a slab stressed in its plane has the form $(\Phi = \mathbf{\Phi}\cdot\mathbf{N})$

$$\delta V = 0, \quad V = \iint_O \{12\,(Eh^3)^{-1}[(1+\nu)\,\mathrm{tr}\,\mathbf{\Psi}^2 - \nu\,\mathrm{tr}^2\,\mathbf{\Psi}] + (\mathbf{e}\cdot\nabla\Phi - \nabla\cdot\mathbf{\Psi})\cdot\sigma\cdot(\mathbf{e}\cdot\nabla\Phi - \nabla\cdot\mathbf{\Psi})\}\,dO \tag{7.1}$$

The stress functions $\Phi, \mathbf{\Psi}$ in the functional V should satisfy the force boundary conditions. The geometric conditions are assumed to be homogeneous. For axisymmetric modes of buckling of a uniformly compressed $(\mathbf{S} = -q\mathbf{G})$ circular plate we have $\Phi = 0, \psi_{12} = 0$, where the subscript 1 corresponds to the radial coordinate r, and the subscript 2 to the angular coordinate. In this case the functional (7.1) takes the form

$$V = 2\pi q^{-1}\int_0^a \left[12\,(Eh^3)^{-1}\,q\,(\psi_{11}^2 + \psi_{22}^2 - 2\nu\psi_{11}\psi_{22}) - \left(\frac{d\psi_{11}}{dr} + \frac{\psi_{11}-\psi_{22}}{r}\right)^2\right] r\,dr \tag{7.2}$$

Here a is the plate radius and q is the magnitude of the compressive force. In the case of a freely supported plate we take the expressions for the bending moments occurring in a plate without initial stresses subjected to a uniform transverse load as the coordinate functions of the Ritz method:

$$\psi_{11} = C\,(3+\nu)\,(a^2 - r^2), \quad \psi_{22} = C\,[(3+\nu)\,a^2 - (1+3\nu)\,r^2] \tag{7.3}$$
$$C = \mathrm{const}$$

From (7.1)—(7.3) we find the approximate value of the critical load for $\nu = 0.3$

$$12(1-\nu^2)\,(Eh^3)^{-1}a^2q = 4.25$$

The exact value of this quantity is 4.20 /12/.

The problem of the stability of compressed thin-walled cylindrical rods of a closed profile might be another example of the application of the theory presented above. For the rod buckling modes of a thin-walled cylinder whose transverse section is a smooth closed contour, the bending stiffness of the shell wall can be neglected, i.e., membrane theory can be considered. According to (4.2), the specific additional energy in this case takes the form (the unimportant constant factor is neglected)

$$A = H_{11}^2 + H_{22}^2 - 2\nu H_{11}H_{22} + \frac{1+\tau}{\tau}(\lambda_2^2 + H_{12}^2 + H_{21}^2) + \frac{2\,(\nu\tau - 1)}{\tau}H_{21}H_{12}, \quad \tau = \frac{T}{E} \tag{7.4}$$
$$H_{\alpha\beta} = \mathbf{e}_\alpha\cdot\mathbf{H}\cdot\mathbf{e}_\beta \quad (\alpha, \beta = 1, 2)$$

Here τ is the dimensionless initial stress acting in the direction of the rod axis. Taking account of (3.2) and (6.2), we express the additional energy in terms of the stress functions Φ_1, Φ_2 and we apply a variational principle with the functional (5.4) to investigate the stability of a rod whose endfaces are under sliding clamping conditions. In this case we should set (l is the rod length)

$$\Phi_1 = \varphi_1(x_1)\sin\eta x_2, \quad \Phi_2 = \varphi_2(x_2)\cos\eta x_2, \quad \eta = \frac{n\pi}{l}, \quad n = 1, 2, \ldots$$

Taking account of the periodicity of the functions $\varphi_\alpha(x_1)$ $(\alpha = 1, 2)$, the condition for the functional (5.4) to be stationary results in the equations (the prime denotes the derivative with respect to x_1)

$$(1+\tau)\varphi_1'' - [(1+\tau)k^2(x_1) + \eta^2\tau]\varphi_1 - \eta\varphi_2' = 0$$
$$\tau\varphi_2'' - (1+\tau)\eta^2\varphi_2 + \eta\varphi_1' = 0 \tag{7.5}$$

Here $k(x_1)$ is the curvature of the cross-sectional contour. Equations (7.5) agree with the equations derived in /13/ by another method that applied the principle of additional energy directly to the three-dimensional theory of equilibrium of prestressed bodies.

The domain of applicability of this theory for the buckling of thin-walled rods is studied in /14/ in the example of a rod with circular section by making a comparison with the exact solution of the stability problem for a hollow circular cylinder in a three-dimensional formulation. It is established in /14/ that Eq.(7.5) enables the critical load to be determined fairly exactly, corresponding to the rod instability mode occurring in long shells. This buckling mode is characterized by the fact that the functions $\varphi_\alpha(x_1)$ have two sign changes on the cross-sectional contour. Equations (7.5) are used in /14/ to calculate the critical laod of a rod with a complex cross-sectional profile.

REFERENCES

1. ZUBOV L.M., Theory of small strains of prestressed thin shells, PMM. Vol.40, No.1, 1976.
2. LUR'E A.I., General theory of elastic thin shells, PMM. Vol.4, No.2, 1940.
3. GOL'DENBERG A.L., Theory of Elastic Thin Shells. Nauka, Moscow, 1976.
4. NOVOZHILOV V.V., Theory of Thin Shells. Sudpromgiz, Leningrad, 1962.
5. CHERNYKH K.F., Linear Shell Theory, Izdat. Leningrad. Gosud. Univ., Pt.1, 1962. Pt.2, 1964.
6. ZUBOV L.M., Static geometric analogy and variational principles in non-linear membrane shell theory. Proceedings XII All-Union Conf. On the Theory of Shells and Plates, Vol.2, Izdat. Erevan Univ., 1980.
7. ZUBOV L.I., Methods of the Non-linear Theory of Elasticity in the Theory of Shells. Izdat. Rostov Univ. Rostov-na-Donu, 1982.
8. DE WITT R., Continuum Theory of Disclinations /Russian translation/, Mir, Moscow, 1977.
9. ZUBOV L.M., Variational principles of the non-linear theory of elasticity. The case of the superposition of a small strain on a finite strain, PMM. Vol.35, No.5, 1971.
10. LUR'E A.I., Elasticity Theory. Nauka, Moscow, 1970.
11. ABOVSKII N.P., ANDREYEV N.P. and DERUGA A.P., Variational Principles of Elasticity Theory and the Theory of Shells. Nauka, Moscow, 1978.
12. TIMOSHENKO S.P., Stability of Elastic Systems /Russian translation/ Gostekhizdat, Moscow, 1955.
13. DIKALOV A.I. and ZUBOV L.M., On the theory of small strains of an elastic solid imposed on a state of uniaxial compression, Izc. Sev.-Kavkaz. Nauch. Tsentra Vyssh. Shkoly, Estestv. Nauk, No.4, 1974.
14. ZUBOV L.M. and RUDENKO G.G., Stability of thin-walled rods of closed profile. Izv. Sev. -Kavkax. Nauch. Tsentra Vyssh. Shkoly. Estestv. Nauk, No.2, 1981.

Translated by M.D.F.

PMM U.S.S.R., Vol.49, No.1, pp.95-100, 1985
Printed in Great Britain

0021-8928/85 $10.00+0.00

THE EXISTENCE OF AN OPTIMAL SOLUTION IN PROBLEMS OF DETERMINING THE SHAPE OF AN ELASTIC LINE*

E.A. NIKOLAEVA and L.V. PETUKHOV

The existence of an optimal solution in the problem of strain energy minimization of maximization for an elastic rod is investigated. It is established that for any elastic line shape a unique solution exists in Timoshenko's theory for the boundary conditions under consideration, while there is a case in Kirchoff's theory for an inextensible rod when the solution is not unique. A generalized optimal control exists in the optimization problem. The case when a measurable optimal control exists is investigated. Examples of the generalized control are presented.

1. Let two points O and x_l be fixed in R^3. Connected them by an elastic line of given length l so that the elastic strain energy is extremal. For this problem the load can be considered to be both distributed $\mathbf{p}(\Gamma)$, $\mathbf{m}(\Gamma)$ (vectors of the forces and moments), and lumped

**Prikl.Matem.Mekhan.*, 49, 1, 130-135, 1985